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Generalized Plane Gravitational Waves

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The definition of plane gravitational waves is generalized to include the case in which rays are not orthogonal to the two-dimensional wave surfaces. All Einstein spaces and some new solutions of the Einstein–Maxwell equations of this type are given.

1. INTRODUCTION

The naive notion of plane gravitational waves—namely, the existence of coordinates in which surfaces of constant $g_{\alpha\beta}$ at constant coordinate time are Euclidean 2-spaces "moving" at fundamental velocity in time—lends itself to an invariant definition of plane gravitational waves according to the symmetry groups of $g_{\alpha\beta}$, which appears to be a valid generalization of the usual definition of plane gravitational waves.

Section 2 contains the definition of these waves. Reasons for interpreting them as waves are discussed in the following sections. In Section 8 a list of exact solutions is given.

The following conventions are used: $i, j, k, \ldots = 1,2,3$; $\alpha, \beta, \gamma, \ldots = 1,2,3,4$; $\operatorname{sign}(g_{\alpha\beta}) = 2$; κ denotes Einstein's gravitational constant.

2. DEFINITION AND CANONICAL FORMS

Definition 2.1. The metric tensor $g_{\alpha\beta}$ of a nonflat four-dimensional Riemannian manifold V_4 of signature 2 defines "generalized plane gravitational waves" if $g_{\alpha\beta}$ satisfies the field equations and if its symmetry group G_r

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contains at least one Abelian three-parameter subgroup $G_3^0 \subseteq G_r$ which acts on hypersurfaces V_3, and has a one-parameter subgroup $G_1 \subset G_3^0$ with lightlike trajectories.

Theorem 2.1 (Kellner, 1975, p. 164). The hypersurfaces V_3 are either pseudo-Euclidean or null, and correspondingly there are two types of such waves which, in "canonical coordinates," take the following forms:

Type I: V_3 null $\quad ds^2 = g_{kl}\,dx^k\,dx^l + 2\,dx^1\,dx^4, \quad |g_{kl}| = 0 = g_{11}$

Type II: V_3 pseudo-Euclidean $\quad ds^2 = g_{kl}\,dx^k\,dx^l + (dx^4)^2, \quad |g_{11}| = 0$

where the g_{kl} are arbitrary functions of x^4 only. In both cases $\{\xi_k^\alpha = \delta_k^\alpha\}$ is a basis of the Lie algebra of generators of G_3^0, ξ_1^α being the generator of G_1. The hypersurfaces V_3 are given by $x^4 = \text{const}$.

It will be noted that fields of type II are stationary. However, as will be shown in the following sections, this does not necessarily exclude their interpretation as waves.

3. RAYS AND WAVE SURFACES

Definition 2.1 is equivalent to the existence of coordinates such that $g_{\alpha\beta} = g_{\alpha\beta}(x^1 - x^4)$, where the x^1 and x^4 coordinate lines are space- and timelike, respectively.

Moreover, $V_1 = \{x^\alpha | dx^2 = dx^3 = 0,\ dx^1 = dx^4\}$ defines a geodesic null congruence so that, according to the naive notion of plane waves, one is tempted to call the V_1 "rays" and the 2-surfaces V_2 given by $V_2 = \{x^\alpha | x^1 - x^4 = \text{const}\}$ of constant $g_{\alpha\beta}$ at constant time x^4 "wave surfaces."

These terms can be defined in an invariant way as follows: the rays are given by the lightlike trajectories V_1 of $G_1 \subset G_3^0$ and the wave surfaces by the two-dimensional trajectories V_2 of the "homogeneity group" $G_2 \subset G_3^0$, $G_1 \not\subset G_2$.

This definition of wave surfaces as two-dimensional trajectories of Abelian two-parameter symmetry groups differs from the usual definition of wave surfaces as spacelike (Euclidean) 2-surfaces orthogonal to the rays (Ehlers and Kundt, 1962, p. 85) but it shall be shown that the former includes the latter as special case and thus allows a more general concept of plane gravitational waves. In particular, as is the case with all fields of type II, the rays need not necessarily be orthogonal to the wave surfaces anymore. The rays always have vanishing shear and expansion whereas the twist as well as the covariant derivative of ξ_1^α does not vanish in general.

(For a definition of these "optical parameters" see Ehlers and Kundt, 1962, p. 58.)

The V_2 defined above can be Euclidean as well as pseudo-Euclidean or null. For generality's sake, all three possibilities shall be admitted and respectively described as Euclidean, pseudo-Euclidean, or null waves.

4. ANALOGY TO PLANE ELECTROMAGNETIC WAVES

Pure ($g_{\alpha\beta}$ satisfies the vacuum field equations) plane gravitational waves are usually defined in analogy to plane electromagnetic waves in flat space-time given in appropriate Lorentz frames by field tensors

$$F_{\alpha\beta}=K_{\alpha\beta}f(x^1-x^4), \qquad K_{\alpha\beta}=-K_{\beta\alpha}\in\mathbb{R}, \qquad K_{\alpha\beta}K^{\alpha\beta}=\overset{*}{K}_{\alpha\beta}K^{\alpha\beta}=0$$

or superpositions of these (Synge, 1956).

Such fields admit a five-parameter symmetry group. In analogy to this property Bondi, Pirani, and Robinson (1959) defined pure plane gravitational waves as vacuum fields admitting a five-parameter group of symmetries.

Another definition is given by Ehlers and Kundt mainly in analogy to the geometrical properties of the rays of plane electromagnetic waves of the above type:

A nonflat vacuum field is a plane wave if and only if it admits an Abelian G_3 with three-dimensional lightlike trajectories containing a lightlike G_1 (Ehlers and Kundt, 1962, p. 95, Theorem 2-5.8) (Incidentally, one need not demand the existence of the G_1, since it can be shown that this follows automatically from the existence of the G_3 with three-dimensional lightlike trajectories and the vacuum field equations.)

One can show that in both cases coordinates exist in which the line element takes the form

$$ds^2=g_{22}(dx^2)^2+g_{33}(dx^3)^2+2g_{23}\,dx^2\,dx^3+2dx^1\,dx^4$$

where the g_{ab}, $a,b=2,3$ are functions of x^4 satisfying the field equations

$$\left(g^{ab}g_{ab|4}\right)_{|4}+\tfrac{1}{2}g^{ab}g_{bc|4}g^{cd}g_{da|4}=0 \qquad a,b,c,d=2,3$$

The two definitions are thus equivalent.

From Theorem 2.1 it follows that these waves represent a subclass of the generalized plane vacuum waves and it is easily seen that this subclass is characterized by the fact that the rays are orthogonal to the wave surfaces—

an obvious consequence of the fact that the rays of the particular electromagnetic waves used as pattern are also orthogonal to the wave surfaces.

There is, however, no particular reason to restrict the analogy to just these special electromagnetic plane waves, and it can be shown that the generalized plane gravitational waves can be put into complete analogy with electromagnetic plane waves once these are defined in a more general and covariant way:

Definition 4.1. The tensor field $F_{\alpha\beta} = -F_{\beta\alpha}$ describes plane electromagnetic waves (in vacuo) if and only if

(i) $F^{\alpha\beta}{}_{;\beta}=0, \qquad \overset{*}{F}{}^{\alpha\beta}{}_{;\beta}=0$
(ii) $I_1=\frac{1}{2}F_{\alpha\beta}F^{\alpha\beta}=0, \qquad I_2=\frac{1}{2}\overset{*}{F}_{\alpha\beta}F^{\alpha\beta}=0$
(iii) $F_{\alpha\beta}$ admits an Abelian three-parameter symmetry group G_3^0 with one- and two-parameter subgroups G_1 and G_2, $G_1 \subset G_2$, whose trajectories are geodesic null lines V_1 and two-dimensional Euclidean, pseudo-Euclidean, or null surfaces V_2, respectively.

Condition (ii) guarantees transport of electromagnetic energy at the speed of light (Pirani, 1956). Condition (iii) implies the existence of rays V_1 and wave surfaces V_2 in the same way as in the case of the generalized plane gravitational waves. For generality's sake, also pseudo-Euclidean and null wave surfaces have again been permitted.

Such waves contain waves of type $F_{\alpha\beta}=K_{\alpha\beta}f(x^1-x^4)$, where the rays are orthogonal to the wave surfaces, as special cases. In general the rays need not be orthogonal. The existence of such fields is shown in Section 8.

From the standpoint of symmetry, the generalized plane gravitational waves are now clearly seen to be in complete analogy to these plane electromagnetic waves.

All generalized pure plane gravitational waves, as well as all other Einstein spaces satisfying Definition 2.1 are listed in Section 8.

5. ASSOCIATION WITH PLANE ELECTROMAGNETIC WAVES VIA THE EINSTEIN-MAXWELL EQUATIONS

In canonical coordinates the Ricci tensor of the waves of type I takes the form

$$R_{\alpha\beta}=R_{44}\underset{1}{\xi}_{\alpha}\underset{1}{\xi}_{\beta}$$

where $\underset{1}{\xi}^{\alpha}=\delta_1^\alpha$ is the lightlike generator of the G_1. In this case the Rainich conditions are satisfied:

$$R=0, \quad R_\rho^\sigma R_\sigma^\beta=\tfrac{1}{4}R_{\mu\nu}R^{\mu\nu}\delta_\rho^\beta$$

which imply (Witten, 1962) the existence of a tensor field

$$F_{\mu\nu} = -F_{\nu\mu}:\ R_\rho^\beta = \kappa\left(F_{\rho\sigma}F^{\sigma\beta} + \tfrac{1}{4}\delta_\rho^\beta F_{\mu\nu}F^{\mu\nu}\right)$$

From this one gets the idea that apart from vacuum fields there might also exist solutions of the Einstein–Maxwell equations of type I. Because of $I_1^2 + I_2^2 = \frac{1}{4}R_{\mu\nu}R^{\mu\nu} = 0$ the corresponding Maxwell fields would represent electromagnetic radiation (Pirani, 1956). Such solutions with plane electromagnetic waves with rays orthogonal to the wave surfaces have been found and discussed by Takeno (1958).

It appears that also certain waves of type II have Ricci tensors of the above form, and the question arises again whether there are solutions of the Einstein–Maxwell equations with these fields. The answer is yes, and solutions of this type with plane electromagnetic waves with rays nonorthogonal to the wave surfaces are given in Section 8.

This shows that also the stationary fields of type II can correspond to transport of energy at the speed of light.

6. LICHNEROWICZ'S CRITERION

In his general definition of gravitational radiation of arbitrary symmetry Lichnerowicz demands the existence of wave fronts (3-surfaces of discontinuity in the second derivatives of the $g_{\alpha\beta}$) and arrives at the conclusion that a nonflat space-time describes gravitational radiation if there exists a lightlike vector η_α satisfying $R_{\alpha\beta} = \tau\eta_\alpha\eta_\beta$ and $\eta_\alpha R^\alpha_{\rho\beta\gamma} = 0$ (Zakharov, 1973). (Vacuum fields of this type have Riemannian tensors of Petrov type N.)

These conditions are satisfied by all generalized plane gravitational waves with rays orthogonal to the wave surfaces as well as by all solutions of type II of the Einstein–Maxwell equations given in Section 8 and by a certain subclass of the vacuum fields of type II as indicated in Section 8. In all cases the lightlike vector η_α is given by $\eta_\alpha = \underset{1}{\xi}_\alpha$.

On the strength of Lichnerowicz's criterion this provides further evidence for the claim that also the stationary fields of type II can describe radiation—even though in this case possible wave fronts S which are given by spaces normal to η_α are not "parallel" to the wave surfaces any more: $S \cap V_2$ can at most be one-dimensional.

7. ACTION ON TEST PARTICLES

If $\underset{\perp}{\xi}^\alpha$ denotes the projection of the lightlike Killing vector $\underset{1}{\xi}^\alpha = \delta_1^\alpha$ into the three-dimensional rest space of an observer moving at 4-velocity u^α

along a timelike geodesic, the longitudinal effect of the gravitational field can obviously be described by the scalar $L = \underset{\perp}{\xi}_\alpha b^\alpha$, where $b^\alpha = R^\alpha_{\ \rho\beta\gamma} u^\rho u^\beta v^\gamma$ gives the acceleration of infinitesimally close test particles relative to the observer, v^α being the vector of geodesic deviation.

From $u_\alpha b^\alpha = 0$ it follows that $L = \underset{\perp}{\xi}_\alpha b^\alpha = \underset{1}{\xi}_\alpha b^\alpha = \underset{1}{\xi}_\alpha R^\alpha_{\ \rho\beta\gamma} u^\rho u^\beta v^\gamma$. Thus all fields mentioned in Section 6 satisfying Lichnerowicz's criterion have $L = 0$ since $\underset{1}{\xi}_\alpha = \eta_\alpha$, and are purely transverse in this sense, whereas in the remaining fields listed in Section 8 longitudinal effects do occur. Since Lichnerowicz's criterion is neither sufficient nor necessary for the existence of wave fronts, such effects are not necessarily excluded by this criterion.

8. EXACT SOLUTIONS

The exact solutions of type I of the vacuum and Einstein–Maxwell equations are given elsewhere (Landau and Lifschitz, 1963; Takeno, 1958); thus only solutions of type II are listed. As noted earlier rays are not orthogonal to the wave surfaces in this case.

Einstein Spaces. For vanishing cosmological constant all solutions have been given by Dautcourt, Papapetrou, and Treder (1962). By suitable coordinate transformations these metrics can be considerably simplified. Together with the solutions with nonvanishing cosmological constant they can be written in compact form as follows (Kellner, 1975, p. 172):

1:

$$ds^2 = X^{2/3}\left[(k_{12}I + k_{22})(dx^2)^2 + (dx^3)^2 + 2k_{12}\,dx^1\,dx^2 + 2k_{23}\,dx^2\,dx^3\right] + (dx^4)^2$$

2:

$$ds^2 = X^{2/3}\left[k_{13}(dx^2)^2 + \left(k_{13}\int \frac{I}{X}dx^4 + k_{23}I + k_{33}\right)(dx^3)^2 + 2k_{13}\,dx^1\,dx^3 + 2(k_{13}I + k_{23})\,dx^2\,dx^3\right] + (dx^4)^2$$

3:

$$ds^2 = X^{2/3}\left[(k_{12}I + k_{22})e^{aI}(dx^2)^2 + e^{-2aI}(dx^3)^2 + 2k_{12}e^{aI}\,dx^1\,dx^2\right] + (dx^4)^2$$

4:

$$ds = X^{2/3}\left[\left(\frac{k_{33}}{(3a)^4}e^{-3aI} + k_{12}I + k_{22}\right)e^{aI}(dx^2)^2 + k_{33}e^{-2aI}(dx^3)^2 + 6ak_{12}e^{aI}dx^1\,dx^2 + 2\frac{k_{33}}{(3a)^2}e^{-2aI}dx^2\,dx^3\right] + (dx^4)^2$$

$a \neq 0$ and k_{mn} are arbitrary constants, $I = \int dx^4/X$ and X is a function of x^4 defined according to whether the cosmological constant λ is greater, equal to, or less than zero:

$\lambda = 0$:

$$X = \alpha x + \beta, \qquad \alpha^2 + \beta^2 \neq 0, \qquad \alpha = \begin{cases} 0, & \text{case 2} \\ \pm\frac{3}{2}a, & \text{cases 3 and 4} \end{cases}$$

Case 1 does not occur here, since then $R_{\alpha\beta\gamma\delta} \equiv 0$.

$\lambda > 0$:

$$X = \alpha e^{\gamma x^4} + \beta e^{-\gamma x^4}, \qquad \alpha^2 + \beta^2 \neq 0, \qquad \gamma = (3\lambda)^{1/2},$$

$$\alpha\beta = \begin{cases} 0, & \text{cases 1 and 2} \\ -\left(\frac{3}{4}a/\gamma\right)^2, & \text{cases 3 and 4} \end{cases}$$

$\lambda < 0$:

$$X = \alpha\cos\gamma x, \qquad \alpha = \left(\tfrac{3}{2}a/\gamma\right)^2, \qquad \gamma = (-3\lambda)^{1/2} \qquad \text{only cases 3 and 4 occur}$$

These solutions are defined in intervals $J \subset \mathbb{R}$ in which $X \neq 0$. In all cases $X \sim (-|g_{\alpha\beta}|)^{1/2}$. $X = 0$ thus implies singularities of the metric. The constants can always be chosen so that $\operatorname{sign}(g_{\alpha\beta}) = 2$ in J. Of these solutions the metrics which satisfy Lichnerowicz's criterion for gravitational radiation (and thus have Riemann tensors of Petrov type N) are given by solution 2 above with $\lambda = 0$.

Solutions of the Einstein–Maxwell Equations. Special solutions of type II satisfying the Einstein–Maxwell equations with electromagnetic radiation are given by (Kellner, 1975, pp. 153)

$$ds^2 = g_{kl}\,dx^k\,dx^l + (dx^4)^2, \qquad g_{11} = 0$$

and

1:

$$g_{1n}=k_{1n}$$

$$g_{22}=-\frac{\kappa}{\beta^2}k_{12}^2(x^4)^2+ak_{12}x^4+k_{22}$$

$$g_{23}=-\frac{\kappa}{\beta^2}k_{12}k_{13}(x^4)^2+ak_{13}x^4+k_{23}$$

$$g_{33}=-\frac{\kappa}{\beta^2}k_{13}^2(x^4)^2+bk_{13}x^4+k_{33}$$

$$k_{mn}, a, b, \beta\in\mathbb{R}, \qquad ak_{13}=bk_{12}, \qquad k_{11}=0, \qquad \beta\neq 0$$

The corresponding Maxwell field is given by $F^{\alpha\beta}=(2/\beta)\delta^{\alpha}_{[1}\delta^{\beta}_{4]}$

2:

$$g_{1n}=k_{1n}, \qquad g_{22}=k_{22}$$

$$g_{23}=(1/\beta)k_{22}x^4+k_{23}$$

$$g_{33}=\left(\frac{1}{2\beta^2}k_{22}-\frac{\kappa}{\beta^2}k_{13}^2\right)(x^4)^2+\left(a+\frac{1}{\beta}k_{23}\right)x^4+k_{33}$$

$$k_{mn}, a, \beta\in\mathbb{R}, \qquad k_{11}=k_{12}=0, \qquad k_{22}>0, \qquad k_{13}\neq 0, \qquad \beta\neq 0$$

The corresponding Maxwell field is the same as in solution 1.

3:

$$g_{1n}=k_{1n}, \qquad g_{22}=k_{22}(x^4)^2$$

$$g_{23}=\frac{\varepsilon}{2\alpha}k_{22}(x^4)^2+k_{23}$$

$$g_{33}=-\kappa\frac{k_{13}^2}{\alpha^2}\ln^2|x^4|+\sigma\ln|x^4|+\frac{\varepsilon}{4\alpha^2}k_{22}(x^4)^2+k_{33}$$

$$k_{mn}, \alpha, \sigma, \varepsilon\in\mathbb{R}, \qquad k_{11}=k_{12}=0, \qquad k_{13}\neq 0,$$

$$k_{22}>0, \qquad \varepsilon=0,1, \qquad \alpha\neq 0$$

The corresponding Maxwell field is given by $F^{\alpha\beta}=(2/\alpha x^4)\delta^{\alpha}_{[1}\delta^{\beta}_{4]}$.

In all cases the constants can be chosen in such a way that $g_{\alpha\beta}$ has the right signature on the whole real axis with the only exception of the singular point $x^4 = 0$ in case 3, where $|g_{\alpha\beta}| = 0$ for any choice of the constants.

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